5/03/14

Lect 1.

Numbers

$\mathbb{N}$ natural numbers.

$\mathbb{Z}$ integers.

$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$.

$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z},\ q \in \mathbb{N} \right\}$

rational numbers

$\mathbb{R}$ real numbers. Not yet defined rigorously.

$\mathbb{C}$ complex numbers

$\mathbb{C} = \{ a + b i \mid a, b \in \mathbb{R} \}$,

"i is $\sqrt{-1}$"

Note. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields.

<u>Def.</u> A <u>field</u> is a set $\mathbb{F}$ with two operations $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ satisfying the following axioms:

1. $(a+b)+c = a+(b+c)$ for all $a, b, c \in \mathbb{F}$.
2. $a+b = b+a$, $a, b \in \mathbb{F}$
3. There exists $0 \in \mathbb{F}$ s.t. $0+a = a+0 = a$.
4. For each $a \in \mathbb{F}$, there is $(-a) \in \mathbb{F}$ s.t. $a+(-a) = (-a)+a = 0$.
5. $(ab)c = a(bc)$.
6. $ab = ba$
7. There exists $1 \in \mathbb{F}$ s.t. $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{F}$.
8. For each $a \overset{\neq 0}{\in} \mathbb{F}$, there is $a^{-1} \in \mathbb{F}$ s.t. $a^{-1}a = aa^{-1} = 1$.
9. $a(b+c) = ab + ac$ for $a, b, c \in \mathbb{F}$.

Lec. 2

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$. These are fields.

Note: We assume $0 \neq 1$.

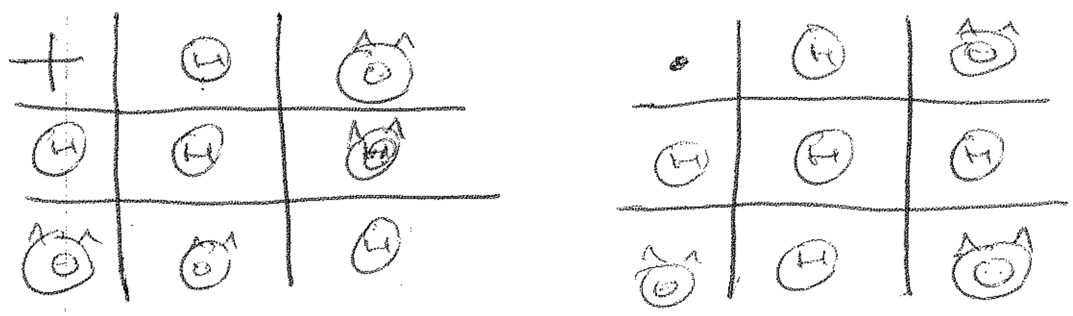
Examples of fields

1) $\mathbb{F} = \{0, 1\}$. How is $+$ defined?

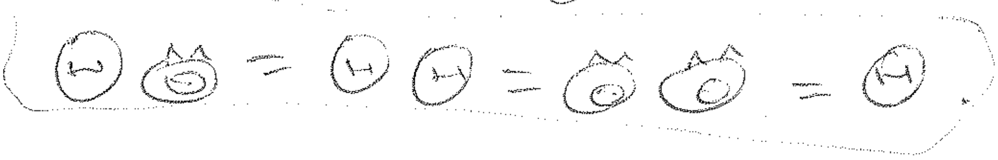
$0+1=1$, $0+0=0$, $1+1=0$,

$0 \cdot 0 = 0$, $0 \cdot 1 = 0$, $1 \cdot 1 = 1$.

Define $+, \cdot$:



What if we define



we have $\underline{0+\tilde{0}}=\tilde{0}$.

But axiom 3 for $\tilde{0}$, we have

$\underline{0+\tilde{0}}=0$. This means $0=\tilde{0}$.

Argument for 1 is the same. $\square$

<u>Thm.</u> If $ab=0$ for $a,b\in\mathbb{F}$, then:

$a=0$ or $b=0$ (or both).

<u>Proof.</u> Assume $a\neq 0$. Then there exist a^{-1} s.t. $a^{-1}\cdot a=1$.

Multiply boths sides of $ab=0$ by a^{-1}.

Then

$$a^{-1}(ab)=a^{-1}\cdot 0.$$

$$a^{-1}(ab)\overset{\text{assoc.}}{=}(a^{-1}a)b=1\cdot b=b.$$

Also, $a^{-1}\cdot 0=0$ by a theorem above.

So $b=0$. $\square$

<u>Theorem.</u> $(-a)(-b)=ab$ for all $a,b\in\mathbb{F}$.

<u>Lemma.</u> $(-a)b=-(a\cdot b)$.

<u>Proof of the lemma.</u> We know

Theorem. For all $a \in \mathbb{F}$,

$$a \cdot 0 = 0 \cdot a = 0.$$

Proof. Axiom 9 for fields implies

$$a(0+0) = a \cdot 0 + a \cdot 0.$$

Also, the properties of 0 imply

$0 + 0 = 0$. Thus,

$$a \cdot 0 + a \cdot 0 = a \cdot 0.$$

Add $-(a \cdot 0)$ to both sides:

$$(a \cdot 0 + a \cdot 0) + (-(a \cdot 0)) = a \cdot 0 + (-(a \cdot 0)),$$

$$a \cdot 0 + (a \cdot 0 + (-(a \cdot 0))) = 0,$$

$$a \cdot 0 + 0 = 0,$$

$a \cdot 0 = 0$. $0 \cdot a = 0$ by commutat. $\square$

Thm. 0 is unique. If $\tilde{0} \in \mathbb{F}$ satisfies axiom 3, then $\tilde{0} = 0$. Also, 1 is unique.

Proof. Because of axiom 3 for 0,

$$(-a)\cdot b + ab \overset{\text{distr.}}{=} (-a+a)b = 0 . b = 0$$

Now, $(-a) b + ab = 0$, add $-(ab)$ to both sides:

$$\big((-a) b + ab\big) + (-(ab)) = 0 + (-(ab))$$

$$(-a) b = -(ab) \qquad \square$$

Proof of the theorem

We know from the lemma that

$$0 \overset{\text{dist}}{=} (-a)(-b) + (-a) b = (-a)(-b) + (-(ab))$$

So $0 = (-a)(-b) + (-(ab))$

Add ab to both sides. Get

$$ab = (-a)(-b) + (-(ab) + ab)$$

$$= (-a)(-b) + 0 = (-a)(-b). \quad \square$$

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3 $\mathbb{Q}$, $\mathbb{R}$ are <u>ordered</u> fields.

LaTeX

ef. $\mathbb{F}$ is an ordered field if
here is a set $P \subset \mathbb{F}$ s.t.

For each $a \in \mathbb{F}$, $a \in P$ or $-a \in P$
or $a = 0$.
(but not at the same time)

If $a, b \in P$, then $a+b \in P$

If $a, b \in P$, then $ab \in P$.

tation. We write $a > 0$ if $a \in P$,
< 0 if $-a \in P$, $a > b$ if $(a-b) > 0$.

eorem. If $x, y \in \mathbb{F}$ and $0 < x < y$,
nd $a > 0$, then $0 < ax < ay$.

of. $ax > 0$ by axiom 3.
ow, to show $ay > ax$, consider

$$y - ax = \underbrace{a}_{>0}\underbrace{(y-x)}_{>0 \text{ since } y > x} > 0. \qquad \square$$

xercises. 1. If $a < 0$ and $b < 0$, then
$ab > 0$.

. If $a < 0$, $b < 0$, then $a+b < 0$.

3. $1 > 0$. Assume $1 < 0$. Then use exercise 1:
$1 \cdot 1 > 0$. But $1 \cdot 1 = 1 < 0$, contradiction. $\square$

4. If $x > 0$, then $x^{-1} > 0$
(we will write $\frac{1}{x}$ for x^{-1}).
Also, if $x > y > 0$, then
$\frac{1}{y} > \frac{1}{x} > 0$.

<u>Theorem.</u> Assume $x, y \geq 0$.
$x > y$ if and only if $x^2 > y^2$.

$$\boxed{x > y} \Longleftrightarrow \boxed{x^2 > y^2}$$

<u>Proof.</u> $\Longrightarrow$ Assume $x > y$. Then
$x^2 = x \cdot x > x y \geq y \cdot y = y^2$.
by the 1st theorem

$\Longleftarrow$ Assume $x^2 > y^2$. Then
$x^2 - y^2 = (x-y)\underline{(x+y)} > 0$. Because

x, y > 0 ⊛, we know (x+y) > 0 by ax. 2.
This means there exists $(x+y)^{-1} > 0$.
Multiply

$$(x-y)(x+y) > 0.$$

~~ply~~ by $(x+y)^{-1}$. This ~~mean~~ yields
$x-y>0$, ~~or~~ which means $x>y$. $\square$

If $x=y=0$, then $x^2=y^2$, so
our assumption doesn't hold.
If $x=0$ and $y>0$, then $x+y=y>0$,
and the same argument applies.
If $x>0$ and $y=0$, same thing.

Some functions. $[0,\infty)$ by def. is $\{a \in \mathbb{R} \mid a \geq 0\}$.

$$|\cdot| : \mathbb{R} \to [0,\infty), \quad |x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

$$[\cdot] : \mathbb{R} \to \mathbb{Z}.$$

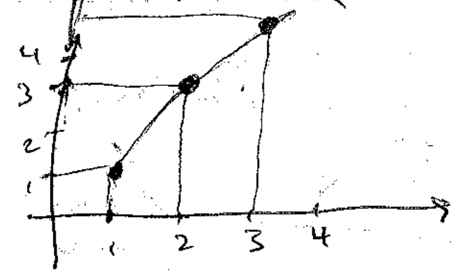
$[x]$ is the greatest integer, which is $\leq x$.

Examples. $[3.8]=3$, $[5]=5$,
$[-2.1]=-3$.

$\sqrt{\cdot} : [0,\infty) \to [0,\infty)$,
$\sqrt{x}$ is the unique nonnegative number y
s.t. $y^2 = x$.

Sequences.

Def. A sequence is a map from $\mathbb{N}$ to some set X.



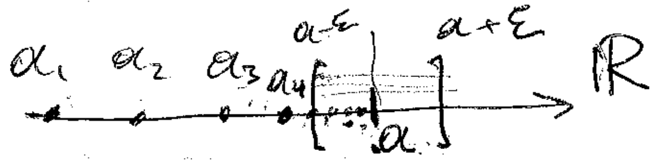
Examples. $1, 3, 5, 7, \ldots$

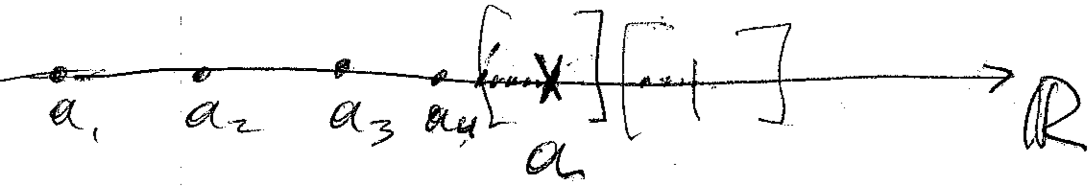
$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

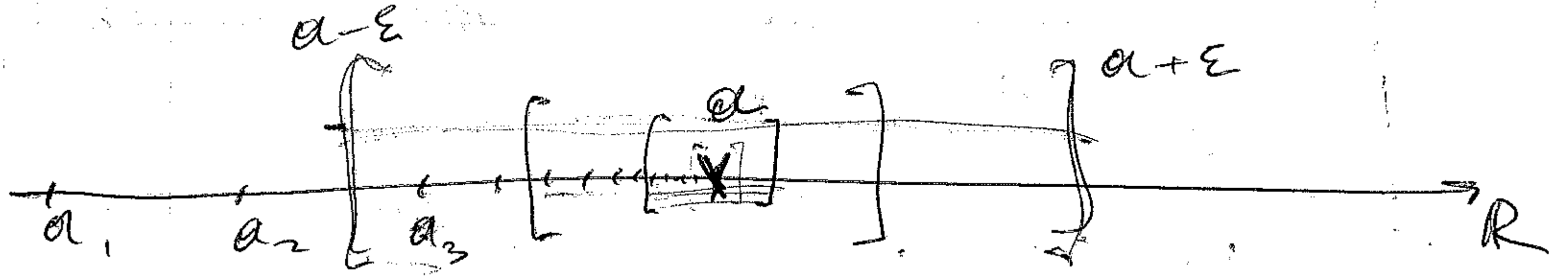
Lec. 4

Def. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. We write $\lim_{n\to\infty} a_n = a$ if for every $\varepsilon > 0$ there exist ~~$\delta>0$~~ $N \in \mathbb{N}$ such that s.t.

$$n \geq N \Rightarrow |a_n - a| < \varepsilon.$$







If $(a_n)_{n=1}^{\infty}$ has a limit, then it <u>converges</u>.
Otherwise, it diverges.

<u>Ex.</u> $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$.

<u>Proof.</u> Fix $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow \left|\frac{n}{n+1} - 1\right| < \varepsilon$.

Compute:

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n - n - 1}{n+1}\right| = \frac{1}{n+1}.$$

We want $\frac{1}{n+1} < \varepsilon$, or

$n+1 > \frac{1}{\varepsilon}$. Take $N = [\frac{1}{\varepsilon}] + 1$.

If $n \geq N = [\frac{1}{\varepsilon}] + 1$, then

$$\left|\frac{n}{n+1}-1\right| = \frac{1}{n+1} \le \frac{1}{[\frac{1}{\varepsilon}]+1+1} \le \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

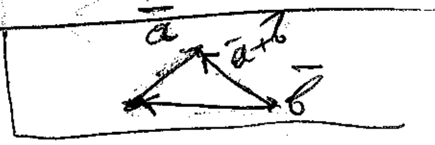
Thus, if $n \geq N = [\frac{1}{\varepsilon}]+1$, then $\left|\frac{n}{n+1}-1\right| < \varepsilon$. This means

$$\lim_{n\to\infty} \frac{n}{n+1} = 1. \qquad \square$$

<u>Note.</u> $|a_n - a| < \varepsilon \iff -\varepsilon < a_n - a < \varepsilon$

$\iff a - \varepsilon < a_n < a + \varepsilon$.

<u>Def.</u> The ε-neighbourhood of a is the set of all $x \in \mathbb{R}$ s.t. $a - \varepsilon < x < a + \varepsilon$.



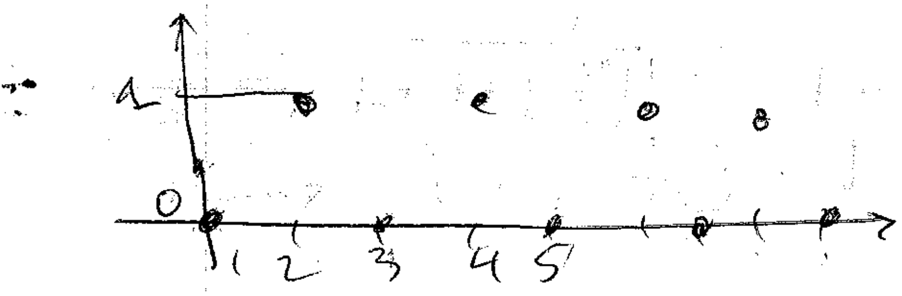
<u>Theorem (triangle inequality).</u> If $a, b \in \mathbb{R}$, then $|a+b| \le |a| + |b|$.

<u>Proof.</u> Method 1. Consider 4 cases.

Method 2. Consider

$$|a+b|^2 = (a+b)^2 = a^2 + b^2 + 2ab$$
$$\le |a|^2 + |b|^2 + 2|a||b| = (|a|+|b|)^2.$$

One of our theorems about ordered fields implies $|a+b| \le |a| + |b|$. $\square$



Divergés.

<u>Theorem</u>. Limit is unique. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$, then $a = b$.

<u>Proof</u>. Fix $\varepsilon > 0$. There exist $N_1 \in \mathbb{N}$ s.t. $n \ge N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$.
Also, there exists $N_2 \in \mathbb{N}$ s.t. if $n \ge N_2$, then $|a_n - b| < \frac{\varepsilon}{2}$.

Set $N = \max\{N_1, N_2\}$.
If $n \ge N$, then $|a_n - a|, |a_n - b| < \frac{\varepsilon}{2}$.
Now

$$|a-b| = |a - a_n + a_n - b|$$
$$\le |a_n - a| + |a_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore for every $\varepsilon > 0$, $|a-b| < \varepsilon$.
This is only possible if $a = b$. $\square$

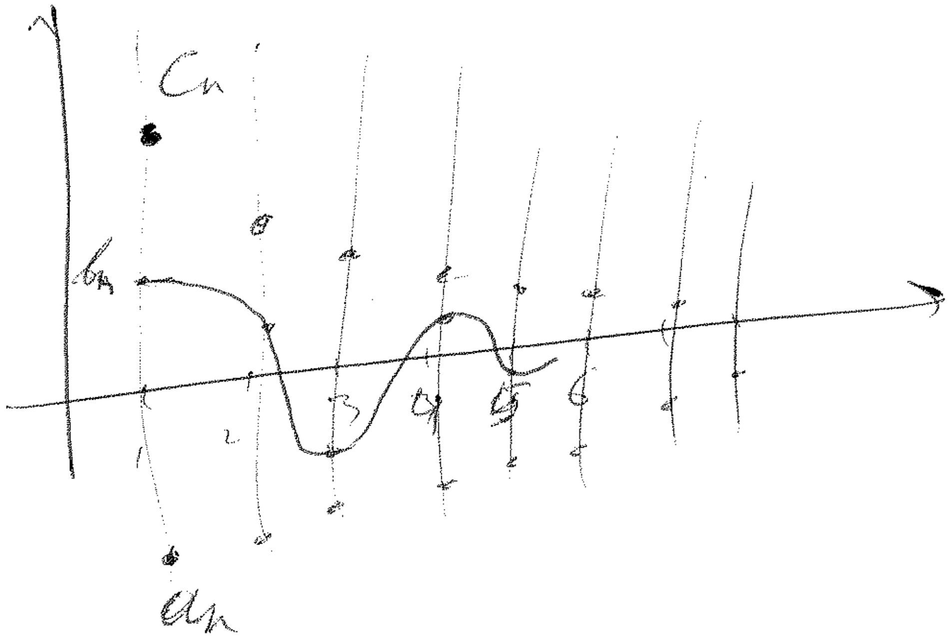
<u>Theorem</u> (Squeeze theorem).

Suppose seqs $(a_n)_{n=1}^{\infty}$ ~~and~~, $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ satisfy

(i) $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$.

Then $\lim b_n = L$.



<u>Proof</u>.